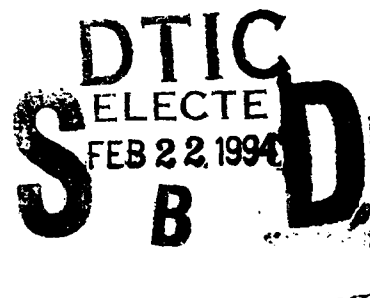AD-A275 990

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UNIFIED ANALYTICAL MODEL OF THE EFFECT OF OBSCURANTS ON TARGET ACQUISITION AND ENGAGEMENT TASKS

Jeffrey F. Nicoll
James D. Silk

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13. ABSTRACT *(Maximum 180 words)*

Models of human search and target acquisition are typically based on the assumption of an independent glimpse probability leading to an exponential model of target acquisition probability. This can be extended to the case of obscurants which intermittently interrupt the line of sight to any point of the image field as might be the case with drifting smokes. The model was previously addressed by Monte Carlo methods by Kowalczyk and Rotman and here is given a complete analytic solution by comparison with physical models of simple ferromagnets. The result is that the single exponential characteristic of the unobscured search model is replaced with two exponentials whose decay times and amplitudes are determined by the average duration times for clear and obscured lines of sight. The modified search model still assumes that the target acquisition task can be decomposed into short glimpses. If a fixed moderate length duration of unobscured time is needed for the completion of a task (such as preparing, firing, and guiding a weapon), a different calculation must be made. Using the obscurant model developed for the search task, the timelines for such situations are given analytically.

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PREFACE

ABSTRACT

Models of human search and target acquisition are typically based on the assumption of an independent glimpse probability leading to an exponential model of target acquisition probability. This can be extended to the case of obscurants which intermittently interrupt the line of sight to any point of the image field as might be the case with drifting smokes. The model was previously addressed by Monte Carlo methods by Kowalczyk and Rotman and here is given a complete analytic solution by comparison with physical models of simple ferromagnets. The result is that the single exponential characteristic of the unobscured search model is replaced with two exponentials whose decay times and amplitudes are determined by the average duration times for clear and obscured lines of sight.

The modified search model still assumes that the target acquisition task can be decomposed into short glimpses. If a fixed moderate length duration of unobscured time is needed for the completion of a task (such as preparing, firing, and guiding a weapon), a different calculation must be made. Using the obscurant model developed for the search task, the timelines for such situations are given analytically.

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EXECUTIVE SUMMARY

The assessment of the effects of smoke on the battlefield presents a uniquely difficult problem for war gamers. In particular, simulations such as JANUS or CASTFOREM play smoke at a level of fidelity such that the players affected by a given smoke round would be identified specifically, but the details of the exact nature of the effect would be treated statistically. In other words, while these simulations are high fidelity, they do not simulate a "virtual reality" in which the details of the smoke plume, such as fluctuations and holes, are represented. The incorporation of such a high level of detail is inconsistent with the design philosophy of these simulation tools. The level of hardware and software reconfiguration that would be required to support such detail would be disruptive to say the least.

Yet the nature of the target acquisition task must be affected by these small scale variations. The existence of fluctuations and holes in smoke plumes has long been known, but has only recently been explained quantitatively. How can these effects be both accurately and economically incorporated into the war games? This is the question which we answer in this paper.

The framework of our analysis is a model of target acquisition in smoke which was developed previously by Rotman. In his treatment, Rotman demonstrates the appropriate averaging procedure for target acquisition in smoke. The analysis herein accomplishes three things. First, we *justify* the model by establishing the connection between it and some new results in the physics of smoke plumes. Second, we *solve* it by deriving simple analytic solution to the model. Third, we *extend* it by showing how the model applies to the computation of the probability of successfully completing a command guidance task.

The justification of the original model addresses two issues. In both cases we show how to determine the inputs required by the Rotman model from the readily measured properties of smoke plumes. The first issue is the structure of the fluctuations in smoke plumes. In discussing this issue we rely on very recent work done by the Army Battlefield Environments Directorate, which applies the theory of atmospheric turbulence to smoke plumes. The second issue is the main simplifying assumption of the model, namely that for purposes of target acquisition, a given line of sight is either clear or obscured.

This observation draws on our everyday experience; smoke plumes seem to have fairly well defined edges, and the fluctuations in concentration are evident to us as actual holes in the plume. We show that this intuitive assessment is borne out by computations based upon the Night Vision Static Model, which shows that under many conditions (which we determine), the predicted target acquisition performance changes rapidly over a narrow range of smoke concentration.

The second accomplishment, namely the solution of the Rotman model, is based on the mathematical similarity of this model to a certain physical model of ferromagnetism. Even given the physical paradigm, though, much original work still needed to be done since the relevant questions for the present case do not map onto the ones asked in the physical problem. So although the arithmetic does sometimes gets thick, this complexity is not gratuitous because the work does not exist elsewhere. Fortunately, the intermediate complexity can be put aside at the end since the final result is quite simple.

Finally, the extension of the model addresses the operational question that follows naturally from the acquisition phase: Given that we have computed the probability of target acquisition, can we use the same framework to compute the probability of maintaining an uninterrupted line of sight to the target for the time that is needed to successfully engage it? The answer is found to be yes, but some approximations need to be introduced if the solution is to be in closed form.

Further study is needed before insertion into the war game simulations should be considered seriously. Apart from obvious concerns over validation, the model should be exercised in a sensitivity analysis, to determine whether, and under what circumstances, the new predictions affect the determination of relevant measures of effectiveness in the war games. This issue continues to be addressed in the Army's Acquisition and Simulation (ACQSIM) Program. Apart from its possible insertion into the war games, the model developed by Rotman and improved here is available as a stand-alone tool for back-of-the-envelope assessments.

I. INTRODUCTION

Rotman[1] has merged a simple model of target acquisition with a simple model of obscuration. He assumes a simple on-or-off model of smoke obscuration, and that the probability of target *not* acquired is given by a decaying exponential in the *unobscured* time on target. The interesting and nontrivial result is that in order to predict the mean time required to acquire a target in the dynamic environment, it is insufficient to simply specify the level of obscuration; it is also necessary to know something about the temporal correlations of the obscured condition. In other words, the effect of smoke clouds that last, say, 2 seconds with 2-second intervals between clouds is much different than those that last 20 seconds with 20-second intervals, even though the average transmission values are the same for the two cases.

Rotman's model results were obtained via Monte Carlo simulation of the obscurant condition. We have found that it is possible to obtain an analytic solution of his model. This has the obvious benefit that it now becomes possible to obtain certain results without recourse to simulations. An additional benefit that accrues may in fact be more important: The methodology which we introduce for handling the smoke model is sufficiently general that we can pose—and answer—questions outside the realm of the simple search model; for example, questions related to the effect of smoke on the subsequent engagement of the target.

In this paper we develop an analytical formalism for the solution of Rotman's model, and apply it to two problems of military interest: modeling the effect of obscurants on rate of acquisition and rate of engagement. In Section II we introduce and justify the assumptions behind Rotman's smoke model, and show how to extract the model parameters for a given obscurant condition. In Section III we write down a formal solution to the model, then show that the problem maps onto a well-known problem in the statistical mechanics of spin systems. The formalism which we use to obtain solutions is somewhat nonstandard, since the quantities that we need to compute are not the ones usually required in the physical problem. We proceed to obtain the analytic solution to our version of the

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1   S.R. Rotman, "Modeling human search and target acquisition performance: II. Simulating multiple observers in dynamic scenarios," *Opt. Eng.* 28 (11) (1989).

problem in Section IV. In Section V we pose and solve a different problem within the context of Rotman's model; namely, computing the probability of maintaining an *uninterrupted*, unobscured line of sight for some fixed time interval. This addresses questions of the success of a subsequent engagement phase in the case, for example, of a command guided munition. The results of Sections IV and V are summarized concisely in Section VI.

II. JUSTIFICATION OF THE MODEL

There are two distinct assumptions which underlie Rotman's approach and which must be addressed before we consider the model in detail. The first is the two-level assumption, the second that there is a characteristic time that governs the transitions between the two levels. In this chapter we consider each of these issues in turn, with special attention to the methods of calculation for the model parameters from operational conditions.

A. THE TWO LEVELS OF OBSCURATION

The model for smoke which Rotman uses is a two-level model. He assumes that there is a certain acquisition rate that is relevant for the "clear" condition, and another slower one for the "smoked" condition. (In this paper we specialize to the case where the slower rate is zero.) This is clearly a simplification of the true situation, where any given level of obscuration is in principle possible. Can a model which makes such an apparently extreme simplification have any relevance at all? We contend that it does. The reason for this is that target acquisition parameters vary appreciably only over a narrow range of obscuration conditions. Most of the time the target visibility is either essentially un-degraded (compared to ambient conditions) or essentially zero. The intuitive justification for this statement comes from the commonplace observation that smoke plumes and clouds seem to have well defined edges, and even holes.

In order to quantitatively justify our claim, we need to discuss the way in which smoke affects target visibility. We do so in the context of the Night Vision Laboratory's Static Model[2] for target acquisition using thermal imaging sensors. According to this model, the rate at which the target is acquired is proportional to the probability of ultimately detecting the target (P_∞), with the constant of proportionality being 0.29 inverse seconds. P_∞ is determined by the ratio of the number of resolvable cycles on target (N) to a cycle criterion constant which is determined by the cognitive level of the acquisition (N_{50}) by a

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[2] J.A. Ratches, W.R. Lawson, L.P. Obert, R.J. Bergemann, T.W. Cassidy, and J.M. Swenson, "Night Vision Laboratory static performance model for thermal viewing systems," Research and Development Tech. Rept. ECOM-7043, U. S. Army Electronics Command, Ft. Monmouth, NJ (April 1975).

target transfer probability function (TTPF). N in turn is computed, in units of the angular target size s, from the apparent thermal contrast of the target (ΔT_a) by applying the inverse of the minimum resolvable temperature (MRT) function, which is the function that quantitatively characterizes the thermal sensor. Finally ΔT_a may be computed from the actual thermal contrast (ΔT) given the attenuation properties of the obscurant. We assume that Beer's Law is valid; that is, that the attenuation is exponential in the concentration length (CL) with extinction coefficient α.

Before translating these words into equations and "turning the crank," it is worthwhile to observe how it is that the continuous variable CL can be transformed into an almost dichotomous target acquisition probability. There are two links in the foregoing chain of relationships that can cause this to happen. First, Beer's Law contains the CL in the exponential, thereby amplifying the effect of small changes in that variable. Second, the TTPF is a threshold function—a "soft" one to be sure, but when combined with the amplification from Beer's Law the effect can be essentially on-off. Presently we will derive the conditions under which this is so.

Summarizing these statements symbolically, we have

$$P_\infty = TTPF(s\ MRT^{-1}(\Delta T\ e^{-\alpha\ CL})/N_{50})\ , \qquad \text{(II-1)}$$

where MRT^{-1} is the functional inverse of the MRT. The TTPF is given in the Model to be

$$TTPF(x) = \frac{x^{2.7+0.7x}}{1+x^{2.7+0.7x}}\ . \qquad \text{(II-2)}$$

In order to get an analytic result, we invoke a widely used exponential approximation to the MRT function[3,4] so that it can be inverted explicitly:

$$MRT(v) = MRT_0\ \exp(\beta_{sys}v) \Rightarrow MRT^{-1}(\Delta T) = \frac{1}{\beta_{sys}}\ln\left[\frac{\Delta T}{MRT_0}\right]\ , \qquad \text{(II-3)}$$

where typical values of the constants MRT_0 and β_{sys} are 4×10^{-3} °C and 6×10^{-4} radians, respectively. Note that in this approximation the target contrast must exceed MRT_0 in order to be detectable.

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[3] R.E. Roberts, "A Simplified Approach to Analyses of Infrared Sensor Performance versus Weather: Theory and Application to the Hanford Data Base (U)," IDA Paper P-1284, October 1977 (CONFIDENTIAL).

[4] R.E. Roberts and L.M. Biberman, "Impact of Atmospheric Propagation on Present and Future-Generation Infrared Imaging Systems: A Reappraisal of 3–5 μm FLIR Performance (U)," IDA Paper P-1284, October 1977 (CONFIDENTIAL).

Given these relations we can compute the acquisition probability as a function of CL for any sensor and scenario conditions we choose. One set of examples is shown in Figure II-1. We have chosen the sensor parameters to be as given above for all cases. The scenario dependent parameters for the Base Case are $\Delta T = 1°$, $d = 2$ milliradians, and $N_{50} = 3.5$ cycles. Each Excursion differs from the Base Case in one parameter, respectively: (1) $\Delta T = 0.5°$, (2) $d = 1$ milliradians, and (3) $N_{50} = 1$ cycle. Our intuitive expectation is confirmed for these cases. We observe the transition from low to high acquisition probability occurring over a narrow range of concentration length.
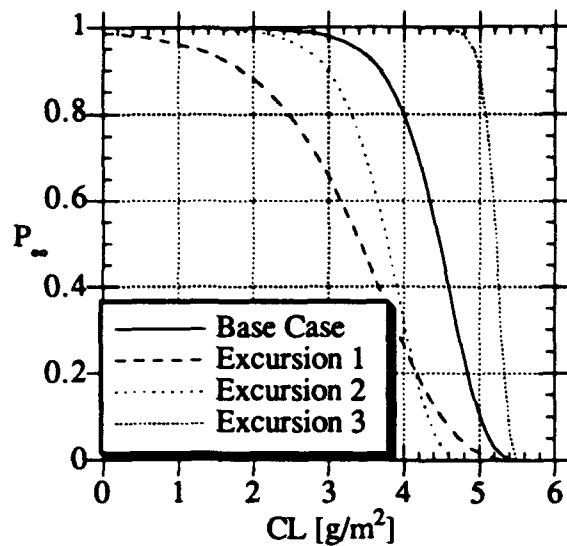
Figure II-1. The value of P_∞ goes from high to low values over a narrow range of CL. The Base Case and the various excursions are described in the text.

In order to elucidate the specific conditions where this situation will occur, some approximations are needed. We estimate the range of CL over which P_∞ varies appreciably to be the inverse of the derivative of P_∞ evaluated at the $P_\infty = 50\%$ point. (Note that this implies that, in the absence of obscurant, the target can be acquired with at least 50% probability.) Taking the derivative of Equation II-1 and using II-2 and II-3 yields

$$\Delta CL = \left[\frac{dP_\infty}{dCL}\Big|_{50\%} \right]^{-1} = 1.18 \frac{\beta_{sys} N_{50}}{\alpha_s} .\qquad (II\text{-}4)$$

The factor 1.18 arises in the derivative of the TTPF. This relation is less simple than it looks. This is because it is not really independent of CL, since we have constrained CL to its 50% value. It is also desirable to remove the "engineering" parameters from the right hand side to the extent feasible. After some straightforward algebra we obtain from Equation II-4 the condition

$$\frac{\Delta CL}{CL_{50}} \approx 1.18 \frac{\ln(\Delta T_{50}/MRT_0)}{\ln(\Delta T/\Delta T_{50})} ,\qquad (II\text{-}5)$$

where CL_{50} is the CL at which the ΔT target is 50% detectable, and ΔT_{50} is the target contrast at which it would be 50% detectable in the absence of obscurants; both definitions

II-3

can be resolved using Equations II-1 to II-3. Now, the target size, sensor resolution, cycle criterion and smoke extinction coefficient all have been absorbed into these two new parameters. Thus our mild approximation has gained us some generality.

Note that, by hypothesis, $\Delta T > \Delta T_{50} > MRT_0$, so both of the logarithms are positive. The ratio of logarithms is smaller than one subject to the condition

$$\Delta T > \frac{(\Delta T_{50})^2}{MRT_0} \quad , \qquad \text{(II-6)}$$

which is the regime in which our two-level approximation is valid; note however that due to the logarithmic figure of merit the two-level hypothesis degrades gracefully.

Given that this condition holds, one needs to be able to extract the necessary model parameters from the smoke data. These are the value of P_{∞} for the unobscured condition, P_{hi}, and the fraction of time that is spent in the unobscured condition, η.

Figure II-2. Illustration of computation of model parameters from plume CL probability density function (PDF) and the P_{∞} function.

Our approach to determining η is illustrated in Figure II-2. We assume a known approximate distribution function, f(CL), which describes the frequency of occurrence of a given concentration length of obscurant. At one level of approximation, we could estimate η by integrating the CL distribution up to the point of inflection of the P_{∞} curve. To account for the fact that our "knife" is a somewhat dull one, we improve our precision by computing the integral

$$\eta = \frac{1}{P_{\infty}(0)} \int_0^{\infty} P_{\infty}(CL)\, f(CL)\, dCL \quad . \qquad \text{(II-7)}$$

The factor in front of the integral corrects for the fact that P_{∞} may not saturate at unity as CL goes to zero. Similarly the appropriate estimate for P_{hi} is given by

$$P_{hi} = \frac{1}{\eta} \frac{1}{P_{\infty}(0)} \int_0^{\infty} \left(P_{\infty}(CL)\right)^2 f(CL)\, dCL \quad . \qquad \text{(II-8)}$$
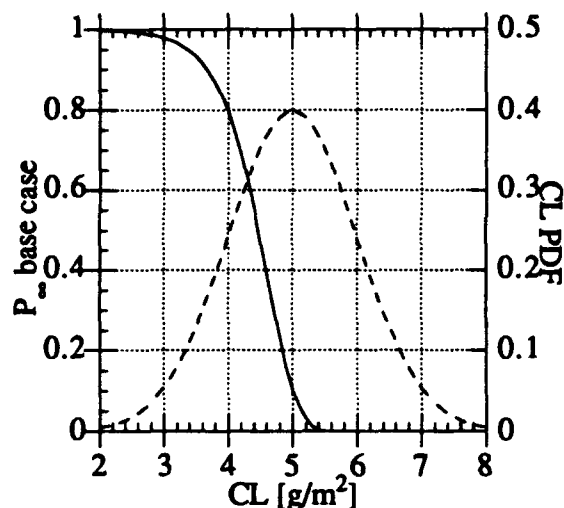
A corresponding P_{low} may be computed (for comparison to zero) by replacing one power of $P_\infty(CL)$ in Equation II-8 by the complementary quantity $\left[P_\infty(0) - P_\infty(CL)\right]$.

B. THE CHARACTERISTIC TIME SCALE

A validated theoretical foundation for the determination of a characteristic time scale for the holes and gaps in smoke plumes has recently become available. Hoock and Sutherland[5] have applied the standard results of Kolmogorov's turbulence theory to the optical transmission problem. They begin by computing the spatial structure function of the CL through a smoke plume. The spatial correlations are then linked to time by invoking a "frozen plume" approximation. This amounts to simply dividing the spatial scale by the transverse component of the ambient wind velocity.

The spatial structure function presented in Hoock and Sutherland is shown as a solid line in Figure II-3. The horizontal axis is length expressed in units of the plume thickness, ℓ. It is interesting that the function exhibits a transition for lengths equal to the thickness of the plume. For shorter lengths, the logarithmic slope approaches 5/3; for larger lengths it approaches 2/3.

This curve actually represents a special case; namely that the "outer length scale," L_0, approaches infinity. This L_0 is the quantity that usually arises in turbulence problems. Roughly speaking, it represents the length scale at which energy is injected into the system. Its value typically is below or around 100 meters. To accommodate finite L_0, we need to go back to the defining equation (Eq. 6.89 on p. 420 of Ref. 4) and numerically reevaluate. The results for particular values of L_0 are shown as dashed and dotted curves in the figure. The net effect is that the curve
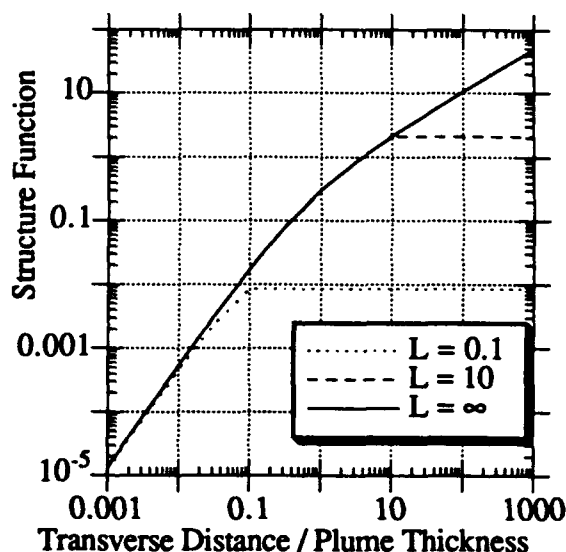
Figure II-3. The smoke plume spatial structure function for various values of L.

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5 Donald W. Hoock and Robert A. Sutherland, "Obscuration Countermeasures," in *The Infrared and Electro-Optical Handbook*, Vol. 7, Countermeasure Systems (David Pollock, ed.) 1993.

is unchanged for small values of separation, but saturates quickly as the separation approaches the outer length scale.

As we shall see later, the temporal correlations in our model are assumed to be exponential. So the parameter we require is the time at which the structure function reaches $(1-1/e) = 0.63$ of its maximum value. Our time scale is therefore of the form

$$T = c \frac{L_0}{V_c} \ , \qquad \text{(II-9)}$$

where c is a dimensionless number of order unity that varies slowly with L_0/ℓ. See Figure II-4. Given that the uncertainties in determining L_0 and V_c will certainly exceed 10%, there is no penalty in assigning c a constant value of 0.55.
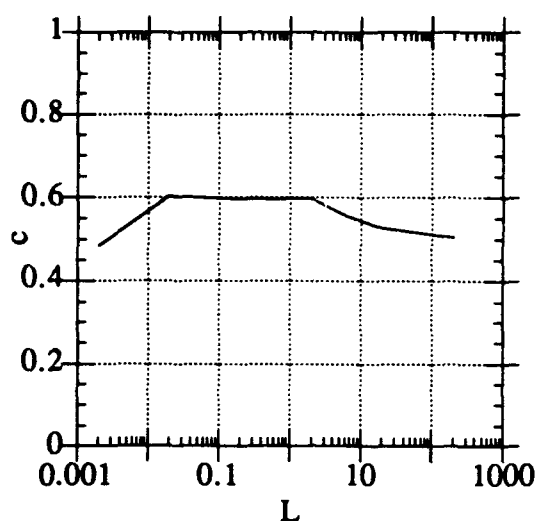
Figure II-4. The L dependence of the "constant" factor c.

III. STATISTICAL MECHANICAL METHOD

Motivated by the discussion in Section II, the probability of an observer who will eventually acquire the target of *not seeing* a target by a time t is assumed to be governed by a differential equation of the form

$$\frac{dP_{fail}}{dt} = -\alpha\, \eta(t)\, P_{fail} \quad , \tag{III-1}$$

where α is the usual target acquisition time constant and $\eta(t)$ is a random function of time describing the ability to see through the obscuration. When the line of sight is completely blocked $\eta = 0$; when the line of sight is clear, $\eta = 1$. Intermediate values model partial seeing; this paper deals primarily with the simpler binary case of complete visibility versus complete blockage; generalizations to partial seeing will be indicated where appropriate. Eq. (III-1) embodies the assumption that the important process in target acquisition is the total time available with a clear line of sight. This is consistent with the modeling of the acquisition process as a series of short glimpses at different parts of the field of view.[6]

The solution to Eq. (III-1) is immediate.

$$P_{fail}(t) = < e^{\displaystyle -\alpha \int_0^t \eta(x)\, dx} > \quad , \tag{III-2}$$

where the brackets (<>) denote an average over the random obscuration $\eta(t)$. In the unobscured limit for which $\eta(t) = 1$ for all values of t, $P_{fail}(t) = \exp(-\alpha t)$ and therefore the probability of successful target acquisition for this observer is $P_{succeed}(t) = 1 - \exp(-\alpha t)$.

This expectation value of an exponential has a strong connection to statistical models of ferromagnetism. To develop this analogy, a "spin" variable s(t) is defined by $\eta(t) = [1 + s(t)]/2$ so that $s(t) = +1$ and -1 correspond to $\eta(t) = 1$, and 0, respectively. Thus, the expectation value in Eq. (III-2) is rewritten as

---

[6] An alternative hypothesis would be to require a fixed amount of continuous uninterrupted observation. This will be addressed in Section V.

$$\langle e^{-\alpha \int_0^t \eta(x)\,dx} \rangle = e^{-\alpha t/2} \langle e^{-(\alpha/2)\int_0^t s(x)dx} \rangle \; . \tag{III-3}$$

It is the average over the spin variable that will be calculated explicitly. In order to apply the standard magnetic models, it is convenient to break up the integral over time into a sum. Divide the interval from 0 to t into N segments of duration Δ, $t = N\Delta$; and rewrite the integral as a sum.

$$\int_0^t s(x)dx = \Delta \sum_{i=0}^{N} s_i \; . \tag{III-4}$$

The large N (equivalently, small Δ) limit will be taken at the end of the calculation.

The problem has now been recast into the form of a one-dimensional chain of spins taking on the values of ± 1 (referred to as "spin up" and "spin down" states); such a model is termed an Ising model. Denoted by $P(\{s\})$ the probability that the set of N+1 spins $\{s\}$ takes on a particular arrangement of up and down spins, the average over spins required for Eq. (III-3) can be written as:

$$\langle e^{-(\alpha\Delta/2)\sum_{i=0}^{N} s_i} \rangle = \sum_{\{s\}} P(\{s\}) \, e^{-(\alpha\Delta/2)\sum_{i=0}^{N} s_i} \; . \tag{III-5}$$

This can be placed into more usual thermodynamic form by representing $P(\{s\})$ in terms of a Hamiltonian or energy function[7]

$$P(\{s\}) \propto e^{-H(\{s\}) + h \sum_{i=0}^{N} s_i} \; , \tag{III-6}$$

where the term linear in the sum of the spin values has been separated from the remainder of the Hamiltonian, $H(\{s\})$. The coefficient of the linear term, h, represents the magnetic field of the spin system and serves to control the average value of the spin. Thus a large positive h will induce the spins to point up; translating back to the obscurant model, a large positive h increases the chance of a clear line of sight.

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[7] To be precise, in a true thermodynamics system this dimensionless Hamiltonian would correspond to the usual Hamiltonian divided by kT where k is Boltzmann's constant and T the absolute temperature. The same is true of the dimensionless Gibbs free energy given in Eq. (III-8).

Inserting this expression for $P(\{s\})$ into Eq. (III-5), the first result of this section is obtained.

$$\left\langle e^{-(\alpha\Delta/2)\sum_{i=0}^{N} s_i} \right\rangle = \frac{\sum_{\{s\}} e^{-H(\{s\}) + h\sum_{i=0}^{N} s_i - (\alpha\Delta/2)\sum_{i=0}^{N} s_i}}{\sum_{\{s\}} e^{-H(\{s\}) + h\sum_{i=0}^{N} s_i}}. \qquad \text{(III-7)}$$

Note that the numerator and denominator of this fraction is of the same form and differ only in the effective value of the "magnetic field," h. In fact, the denominator is just the *partition function* for this system, $Z(h)$, and the numerator is the same partition function with a modified argument, $Z(h - \alpha\Delta/2)$. Using the fact that the Gibbs free energy, G, of a system is related to the partition function by $Z = \exp(-G)$, Eq. (III-7) can be rewritten as

$$\left\langle e^{-(\alpha\Delta/2)\sum_{i=0}^{N} s_i} \right\rangle = \frac{Z(h - \alpha\Delta/2)}{Z(h)} = e^{G(h) - G(h - \alpha\Delta/2)}. \qquad \text{(III-8)}$$

The original problem of the expected value of the probability of not seeing a target in the presence of obscurants is thus shown to be equivalent to finding the Gibbs free energy of a related Ising magnetic system. This "smoke scholium" allows the application of a wide range of one-dimension studies of Ising systems that have been solved for a number of different Hamiltonians. Each choice of Hamiltonian corresponds to a different distribution of smoke properties; a specific choice will have to be made to produce a complete description of the acquisition problem.

Some results can be obtained directly from the thermodynamic formalism and can be used to check the results of any detailed calculation.

(1) For $\alpha \ll 1$, $G(h) - G(h - \alpha\Delta/2) \approx \alpha\Delta/2 N\langle s\rangle$. This is equivalent to:

$$\left\langle e^{-\alpha \int_0^t \eta(x)dx} \right\rangle \approx e^{-\alpha\langle\eta\rangle t}. \qquad \text{(III-9a)}$$

That is, for small α, the time constant is reduced by the average probability of obscuration.

(2) For $t \ll T_c$, where T_c is the smoke correlation time.

$$\left\langle e^{-\alpha \int_0^t \eta(x)dx} \right\rangle \approx p_+ e^{-\alpha t} + p_-, \qquad \text{(III-9b)}$$

III-3

where p_+ and p_- are the probabilities that the initial state of the system was initially (at $t = 0$) in the spin up (clear vision) state and spin down (obscured vision) state, respectively.

(3) For t or N very large, the Gibbs free energy should be "intensive"; that is, G should be proportional to N. In that case,

$$\left\langle e^{-\alpha \int_0^t \eta(x)dx} \right\rangle \approx C\, e^{-\alpha_{eff}\, t}, \tag{III-9c}$$

where C and α_{eff} are constants.

In Section IV, a particular form of the Hamiltonian will be chosen that allows for a simple calculation of the complete result.

IV. SMOKE MODEL RESULTS

The complete solution to Eq. (III-2) can be obtained easily if a simple form of the Hamiltonian is chosen. The one chosen is the simplest Hamiltonian to solve and also corresponds (as will be shown) to the assumption of the Kowalczyk-Rotman Monte Carlo approach. The nearest-neighbor Ising model contains interactions only between adjacent spins:

$$H = -\varepsilon \sum_{i=j\pm1} [s_i s_j - 1] \qquad \text{(IV-1)}$$

where ε is an "interaction energy" and serves to control the coupling or correlation between spins. A large value of this interaction will produce a strong correlation between spins corresponding to longer obscurant correlation times. In this section the partition function ratio for the Ising model Hamiltonian will first be calculated using conventional statistical mechanical techniques. Subsection A will review the solution methods for the usual Ising model; subsection B will derive the results for the partition function ratio which describes the probability of not seeing, Eq. (III-8). The result shows that the choice of the Ising Hamiltonian is equivalent to assuming exponential distributions of both the clear and block intervals. This being the case, a Markov random field approach is appropriate. The rederivation of the results using the Markov approach will be given in subsection C.

A. ISING MODEL SOLUTION

The Ising model Hamiltonian, Eq. (IV-1), is solved conventionally by introducing the transfer matrix, $T_{ij} = \exp(-H_{ij})$ where i and j indicate the states (+1 or −1) of adjacent spins. Including the magnetic field contribution, T is given by

$$T = \begin{vmatrix} e^h & e^{-2\varepsilon} \\ e^{-2\varepsilon} & e^{-h} \end{vmatrix} . \qquad \text{(IV-2)}$$

Now imagine M Ising spins arranged not in a straight chain but in a circle so that the last spin and first spin are identical ($s_1 = s_M$). Then the partition function which is the sum over all states can be represented by

$$Z = \text{trace } T^M . \qquad \text{(IV-3)}$$

The matrix product takes care of the sum over the states of the spins and the trace is the representation of the periodic boundary condition imposed.[8] The trace can be evaluated by recognizing that T can be diagonalized with some orthogonal matrix, $T = ODO^{-1}$, where

$$D = \begin{vmatrix} \lambda_+ & 0 \\ 0 & \lambda_- \end{vmatrix} \qquad \text{(IV-4)}$$

is the matrix of eigenvalues of T ($\lambda_+ > \lambda_-$):

$$\lambda_\pm = \cosh(h) \pm \sqrt{\sinh^2(h) + e^{-4\varepsilon}} \qquad \text{(IV-5a)}$$

and

$$O = \begin{vmatrix} \cos\Theta & \sin\Theta \\ -\sin\Theta & \cos\Theta \end{vmatrix} \qquad \text{(IV-5b)}$$

where Θ is the rotation angle of the orthogonal matrix. Then the partition function is

$$Z = \lambda_+^M \left[1 + \left(\frac{\lambda_-}{\lambda_+}\right)^M \right]$$

$$\approx \lambda_+^M, \text{ as } M \to \infty \qquad . \qquad \text{(IV-6)}$$

A number of quantities can be calculated in addition to the partition function. The average value of the spin <s> is given by

$$<s> = \cos^2\Theta - \sin^2\Theta = \frac{\sinh(h)}{\sqrt{\sinh^2(h) + e^{-4\varepsilon}}} \qquad . \qquad \text{(IV-7)}$$

The probabilities of a particular site being spin up (clear line of sight) or spin down (blocked line of sight) were denoted as p_+ and p_- in Sec. III and are given by $p_+ = \cos^2\Theta$ and $p_- = \sin^2\Theta$, respectively. Finally, the spin-spin correlation function is given by

$$<s_i \, s_j> = <s>^2 + (1 - <s>^2) \left(\frac{\lambda_-}{\lambda_+}\right)^{|i-j|} \qquad . \qquad \text{(IV-8)}$$

---

[8] The matrix T^M itself is a matrix of partition functions for fixed boundary conditions of the open chain of M+1 spins, the ++ element of which corresponds to the partition function with s_0 and s_M both up, etc.

IV-2

This completes the solution of the usual Ising model for the infinite chain,[9] but is not sufficient to evaluate the ratio of partition functions in Eq. (III-8) that require consideration of finite chain effects. For future use, the appropriate limits of the infinite chain Ising quantities for the small Δ limit are needed. Consider a scaled magnetic field $h = h\,\Delta$, and near-neighbor interaction parameter, $\exp(-2\varepsilon) = \varepsilon\,\Delta$. Then

$$\lambda_{\pm} = 1 \pm \Delta\beta/2$$

$$\beta/2 = \sqrt{h^2 + \varepsilon^2}$$

$$<s> = \frac{2h}{\beta} \tag{IV-9}$$

$$<s(o)s(\tau)> = <s>^2 + (1 - <s>^2)\,e^{-\beta|\tau|} \;.$$

These changes of scale were chosen to hold the average value of the spin, $<s>$ (related to the average probability of obscurant), and the correlation time of the obscurant (given by $1/\beta$) fixed in the limit $\Delta \to 0$.

B. APPLICATION TO THE SMOKE SCHOLIUM

Instead of a circle of M spins with a single interaction between all the spin pairs, imagine that M-N spins interact with the Ising Hamiltonian and magnetic field h with transfer matrix T, while the remaining spins have a field $h - \alpha\Delta/2$ and transfer matrix, T_α. Then the partition function for this case, denoted $Z(\alpha)$, is simply

$$Z(\alpha) = \text{trace}\,(T^{M-N}\,T_\alpha^N) \;. \tag{IV-10}$$

The desired ratio of partition functions in Eq. (III-8) is $Z(\alpha)/Z(0)$. Denoting by D and D_α the eigenvalue matrices for the two cases, and O and O_α the corresponding rotation matrices, this is:

$$Z(\alpha) = \text{trace}\,(D^{M-N}O^{-1}O_\alpha D_\alpha^{\,N}O_\alpha^{-1}O) \;. \tag{IV-11}$$

The product of the two rotation matrices is itself a rotation matrix with angle $\delta\Theta = \Theta - \Theta_\alpha$. Taking the large M and large N limits (with $N\Delta = t$ fixed), the partition function can be straightforwardly evaluated. Taking the ratio of $Z(\alpha)/Z(0)$ and replacing the factor of $\exp(-\alpha t)$, the final expression for Eq. (III-2) is

---

[9] It is easy to show that the use of periodic boundary conditions is not essential in the solution.

$$\langle e^{-\alpha \int_0^t \eta(x)dx} \rangle = e^{-(\alpha - \beta_\alpha + \beta)t/2} [\cos^2\delta\Theta + \sin^2\delta\Theta \; e^{-\beta_\alpha t}] \; , \qquad \text{(IV-12a)}$$

where

$$\beta_\alpha^2 = (\beta - \alpha\langle s\rangle)^2 + \alpha^2 (1 - \langle s\rangle^2)$$

$$\cos^2\delta\theta = \frac{1 + \dfrac{\beta - \alpha\langle s\rangle}{\beta_\alpha}}{2} \qquad \text{(IV-12b)}$$

$$\sin^2\delta\theta = \frac{1 - \dfrac{\beta - \alpha\langle s\rangle}{\beta_\alpha}}{2} \; .$$

The result is a mixture of two exponentials whose time constants are functions of the target acquisition time constant, α^{-1}, the smoke correlation time β^{-1}, and the average probability of clear line of sight, $\langle\eta\rangle$. $\langle\eta\rangle$ is given by $(1 + \langle s\rangle)/2$. Figure IV-1 shows a comparison of the line of sight probabilities for a number of cases. The figure shows the unobscured result ($\langle s\rangle = 1$) and two variants of the obscured cases for $\langle s\rangle = 0$ and $\langle s\rangle = -0.5$ (corresponding to $\langle\eta\rangle = 1$, $\langle\eta\rangle = 0.5$, and $\langle\eta\rangle = 0.25$, respectively) and a range of values of β/α: $\beta/\alpha = 0.5, 1.0, 2.0,$ and 5.0. Recall that $\langle\eta\rangle$ is the average probability of a clear line of sight while β measures the scale of the clouds and gaps (large β being equivalent to short intervals of clouds and short gaps for fixed $\langle\eta\rangle$).

Note that the curves for each value of $\langle\eta\rangle$ are bunched together even though the values of β/α vary by a factor of 25. For a fixed value of $\langle\eta\rangle$, the probabilities are bounded by the $\beta = 0$ and $\beta = \infty$ limits:

$$P_{succeed}(t, \beta, \langle\eta\rangle) \geq P_{succeed}(t, 0, \langle\eta\rangle) = \langle\eta\rangle(1 - e^{-\alpha t})$$

$$P_{succeed}(t, \beta, \langle\eta\rangle) \leq P_{succeed}(t, \infty, \langle\eta\rangle) = 1 - e^{-\langle\eta\rangle\alpha t} \qquad \text{(IV-13)}$$

which agrees with Eqs. (III-9a)–(III-9b) (identifying $p_+ = \langle\eta\rangle$ and $p_- = 1 - \langle\eta\rangle$). The relationship between the exact result and the bounds[10] is illustrated in Fig IV-2, which reproduces the $\langle\eta\rangle = 0.5$ cases shown in Fig. IV-1 together with the bounds.

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[10] In fact, the $\beta = 0$ limit is a correct approximation to second order in t; β dependence only enters for t^3 and higher terms.
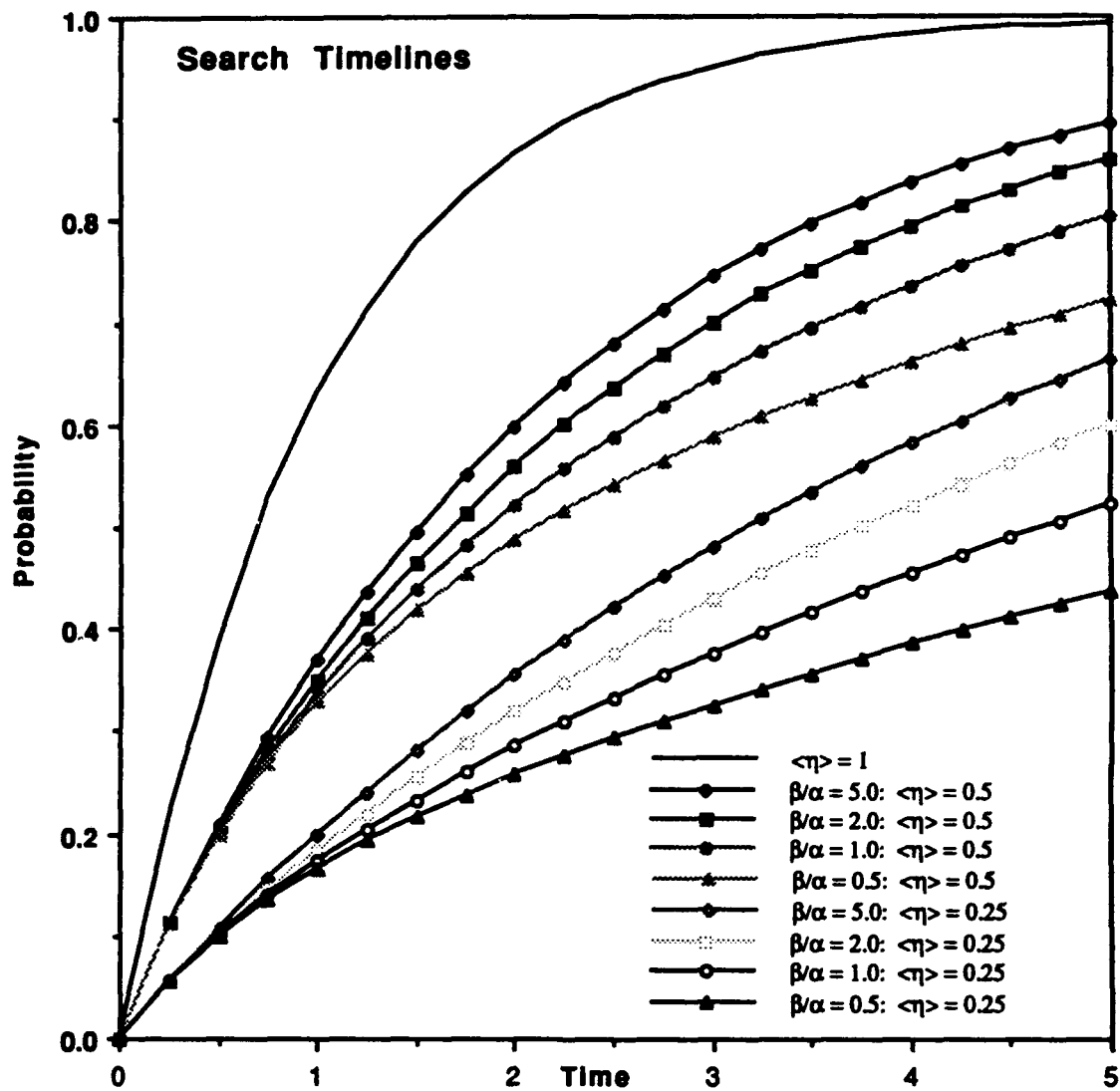
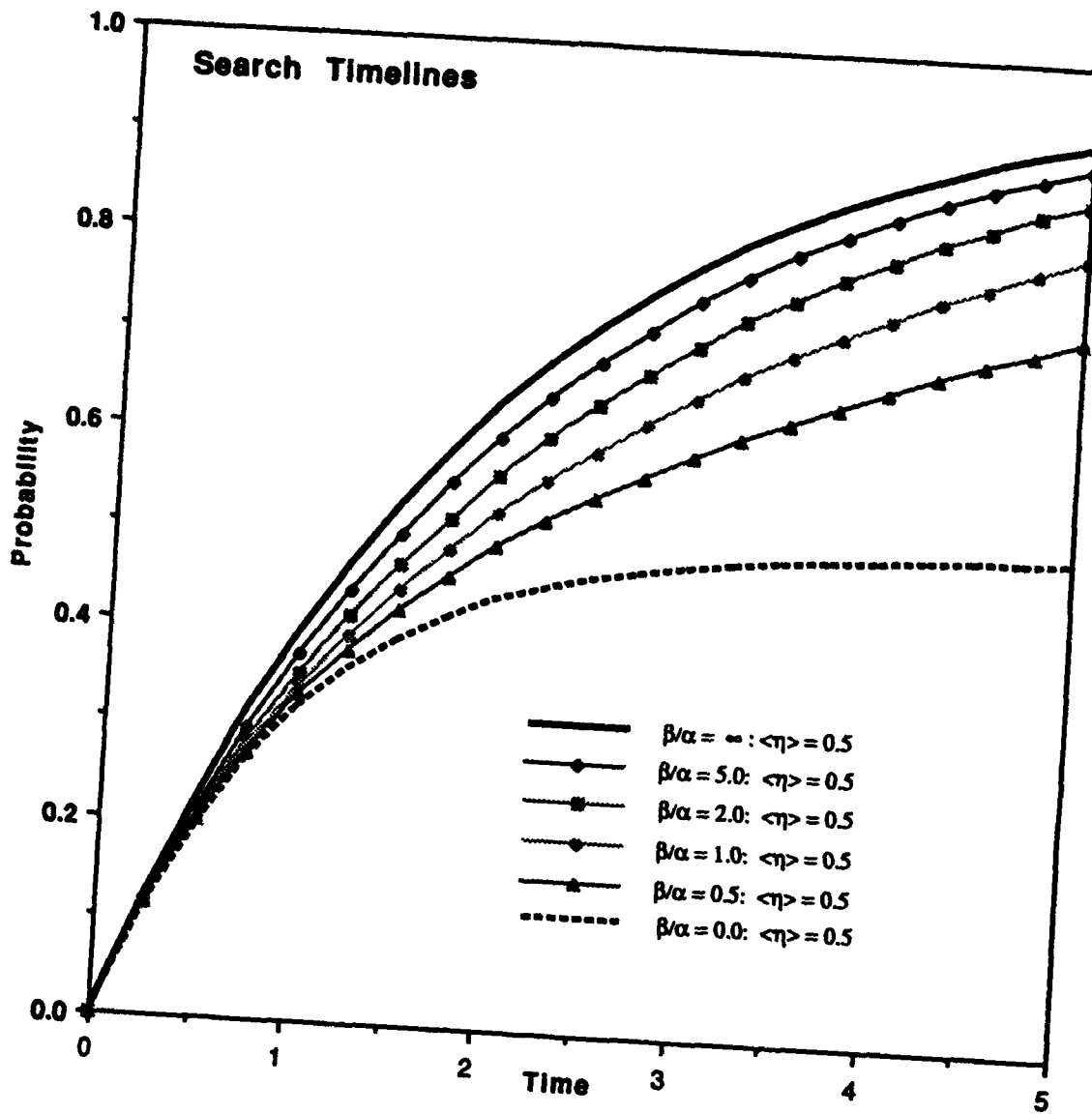**Figure IV-1. Search probabilities as a function of α t for unobscured and obscured cases.**

Figure IV-2. Search probabilities as a function of αt for unobscured and obscured cases. Comparison of exact results for $<\eta> = 0.5$ with approximate bounds.

Finally, for t large, the last limit described in (III-9) can be recovered:

$$\left\langle e^{-\alpha \int_0^t \eta(x)dx} \right\rangle = \cos^2\delta\Theta \, e^{-(\alpha - \beta_\alpha + \beta)\, t/2} \tag{IV-14b}$$

so that $\alpha_{eff} = (\alpha - \beta_\alpha + \beta)/2$.

The final result, Eq. (IV-12), reproduces analytically the Monte Carlo results of Kowalczyk-Rotman. The essential similarity connection is that Kowalczyk-Rotman assume that the smoke obscurations and the spaces between them both have exponential distributions. In this Ising approach, one requires the probability that if a spin at some time, say $t = 0$, is up, that all the spins are up for times between 0 and t. This can be calculated using the transfer matrix approach by inserting into the trace of Eq. (IV-3) the product of n projection operators ($n \, \Delta = t$) of the form

$$P_+ = \begin{vmatrix} 1 & 0 \\ 0 & 0 \end{vmatrix}$$

and then taking the limit of small Δ. Similarly, using the complementary projection operator $P_- = 1 - P_+$, one obtains the probability of staying in a spin down state (line of sight blocked). The resulting probabilities are exponentials:

$$P_\pm (t) = e^{-\beta_\pm t} \tag{IV-15a}$$

$$\beta_+ = \beta \, \frac{(1 - \langle s \rangle)}{2} \tag{IV-15b}$$

$$\beta_- = \beta \, \frac{(1 + \langle s \rangle)}{2} \quad .$$

The two exponents $\beta_\pm$ could be used to characterize the system instead of $\langle \eta \rangle$ and β.

C. MARKOV RANDOM FIELD APPROACH

The Ising model Hamiltonian is therefore equivalent to a two exponent description of the intervals of blocked and clear line of sight. There is an alternative derivation of the result given in Eq. (IV-12) which exploits that relationship directly. This uses, instead of the statistical mechanical analogy, the language of Markov processes.[11] To apply the

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[11] This method was suggested to the authors by Dr. Amnon Dalcher of IDA.

Markov method, consider a three-state system: (1) the spin down state corresponding to blocked sight; (2) the spin up state (clear line of sight); and (3) a "done" state, corresponding to having acquired the target. A transition matrix between these states can be written down immediately in terms of α and β_+. Normally, a three-state system would require a 3×3 transition matrix for a complete description; however, in this case, the done state has no transitions to either the up or down states and therefore a transfer matrix to describe just the transition of the up and down states suffices.[12]

$$ T = \left| \begin{array}{cc} -(\beta_+ + \alpha) & \beta_+ \\ \beta_- & -\beta_- \end{array} \right| \tag{IV-16} $$

Thus, the up state decays through two channels: (1) at a rate β_+ into the down channel and (2) at a rate of α into the done channel. The down state can only transition into the up channel and does so at a rate of β_-. The exponential of T gives the transition probabilities:

$$ e^{Tt} = \frac{1}{\beta_\alpha} \left[e^{\lambda^+ t}(T - \lambda^-) - e^{\lambda^- t}(T - \lambda^+) \right] \tag{IV-17} $$

where $\lambda^\pm$ are the eigenvalues used in Eq. (IV-12):

$$ \lambda^\pm = -\frac{(\beta + \alpha)}{2} \pm \frac{\beta_\alpha}{2} \quad . \tag{IV-18} $$

This can be used to describe the probability of being in the up and down states for any initial distribution of states. The probability of not seeing the target is just the probability of being in either the up or down state (since otherwise the system is in the done state). For any initial distribution of up and down states (v_+, v_-), with $v_+ + v_- = 1$, the probability of not seeing is therefore[13]:

$$ P_{fail}(t) = \frac{1}{\beta_\alpha} \left[e^{\lambda^+ t}(-\lambda^- - v_+\alpha) - e^{\lambda^- \tau}(-\lambda^+ - v_+\alpha) \right] \quad . \tag{IV-19} $$

For the a priori estimate of the distribution of states, $v_+ = p_+ = (1 + <s>)/2$, this reduces to Eq. (IV-12). On the other hand, if one knows the state at $t = 0$ to be definitely clear or blocked the appropriate choice of v_+ is 1 or 0 respectively.

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[12] This means that the transition matrix given here does not "conserve probability." The lost probability corresponds to populating the done state.

[13] This result is more general than that given in Eq. (IV-12) using the statistical mechanical formulation; however, the same general result can be obtained in the partition function approach by using fixed rather than free boundary conditions.

Generalizations of either the partition function approach or the Markov approach could involve the introduction of intermediate states between clear and totally blocked lines of sight. In addition, more complex interactions can be modeled in the Hamiltonian to provide something other than simple exponential distributions of the clear and blocked intervals. However, any of these generalizations involves the introduction of additional parameters. The experimental data available at this time does not indicate the need for any such extension.

V. MODELING EXTENDED INTERVALS

A. APPLICATION OF THE ISING-MARKOV MODELS TO TIME INTERVALS

In the preceding sections, the target acquisition task was modeled as a cumulative process for which the total time available on target was the relevant quantity, independent of how that time was accumulated. This is consonant with the assumption that target acquisition is the consequence of a series of independent glimpses, each of which is of short duration compared to any change in the visibility conditions. Thus, the task is performed equally well when 10 seconds of clear viewing is provided whether it is provided in a single interval of 10 second or in 10 intervals of 1 second each. Another point of view would be to assume that a single continuous block of time is necessary to perform a task. This would be the case, for example, for a command-guided missile, but may also be relevant for other aspects of the target recognition process.

One can describe this problem using the Ising-Markov models of the obscurant developed in Sections III and IV. The line of sight is alternately clear and blocked. It is assumed that the task is completed when a clear line of sight is made available for a time greater than or equal to the required time for the task, τ. As before, it is easy to calculate the probability of failing, that is, the probability as a function of time, t, that no such interval of length τ or greater exists.

Assume for the moment that at $t = 0$, the system is known to be in the spin up (clear) state. Consider a series of flips from spin up to spin down and back again occurring at intervals of t_0, t_1, t_2,...... The total time elapsed, t, is the sum of these intervals.

$$\sum_i t_i = t \ . \tag{V-1}$$

To each transition between states, one can associate a transition probability density of $e^{-\beta_\pm t_j}$ $\beta_\pm dt_j$ where t_j is the length of the interval and the upper sign is used for the transition from an up state and the lower for a transition from a down state. Consider, for example a sequence including M flips, starting initially in the up (+) state. Then the probability density that the M flips representing intervals { t_k } will occur in the total time t is given by

V-1

$$P(M,t,+) = \left(\frac{\beta_+}{\beta_-}\right)^{M\,mod\,2} \prod_{i=0}^{[M/2]} \beta_+\, e^{-\beta_+ t_{2i}}\, dt_{2i} \prod_{j=1}^{M-[M/2]} \beta_-\, e^{-\beta_- t_{2j-1}}\, dt_{2j-1}\, \frac{\delta(\sum_k t_k - t)}{\beta_+} \qquad \text{(V-2)}$$

where [M/2] is the integer part of M/2. The corresponding probability is given by integrating over the allowed values of the sizes of the intervals. Suppose that probability that no (+) state lasts longer than time τ_+ and that no minus (–) state lasts longer that τ_-. Then the integrals may be performed if the delta function is replaced by its Fourier transform:

$$P(M,t,+,\tau_+,\tau_-) = \int \frac{d\omega}{2\pi\beta_+} \left(\frac{\beta_+}{\beta_-}\right)^{M\,mod\,2} R_+^{[M/2]+1}\, R_-^{M-[M/2]} \qquad \text{(V-3a)}$$

where the factors $R_\pm$ are given by:

$$R_\pm = \frac{\beta_\pm}{\beta_\pm + i\omega}\left(1 - e^{-(\beta_\pm + i\omega)\tau_\pm}\right) . \qquad \text{(V-3b)}$$

Summing over all values of M, one has

$$P(t,+,\tau_+,\tau_-) = \int \frac{d\omega}{2\pi} \frac{e^{i\omega t}}{\beta_+ \beta_-} \frac{R_+ R_-}{(1 - R_+ R_-)}\, [\beta_+ + \beta_- R_-^{-1}] . \qquad \text{(V-4)}$$

The corresponding formula for beginning in the down state is given by interchanging plus and minus subscripts. (If the initial state is not known, then one can sum over the two cases using the a priori estimates of being in the up and down states $p_+ = \beta_- /\beta$, $p_- = \beta_+ /\beta$, where, as before, $\beta = \beta_+ + \beta_-$.)

In the case of interest, the amount of time in the down or (–) state is unimportant ($\tau_- = \infty$); writing $\tau_+ = \tau$ for simplicity, Eq. (V-4) then gives the probability that in a time t, beginning in the (+) state, no interval of length τ or longer exists (that is, the probability of failure), as follows:

$$P(t,+,\tau) = \int \frac{d\omega}{2\pi i}\, e^{i\omega t}\, \frac{(\omega - i\beta)(1 - e^{-(\beta_+ + i\omega)\tau})}{\omega(\omega - i\beta) - \beta_+ \beta_- e^{-(\beta_+ + i\omega)\tau}} . \qquad \text{(V-5a)}$$

For starting in the down (–) state:

$$P(t,-,\tau) = \int \frac{d\omega}{2\pi i}\, e^{i\omega t}\, \frac{\omega - i\beta + i\beta_- e^{-(\beta_+ + i\omega)\tau}}{\omega(\omega - i\beta) - \beta_+ \beta_- e^{-(\beta_+ + i\omega)\tau}} . \qquad \text{(V-5b)}$$

The contour integrals in Eq. (V-5) run along and just below the real ω axis. Although these integrals provide the exact results needed, the complicated form of the denominators in these integrals makes them difficult to evaluate. One approach is to expand the denominators in powers of the exponential terms and integrate each term separately. Note that a term proportional to the nth power of the exponential carries an ω dependence of $e^{i\omega(t - n\tau)}$; for $t < n\tau$, the contour can be closed down in the complex ω plane and there is no contribution to the integral. For $t > n\tau$, the contour is closed up and the contributions from the poles at $\omega = 0$ and $\omega = i\beta$ can be easily calculated.[14] Straight-forward, but extremely tedious, algebra gives:

$$P(t, +, \tau) = \sum_j \left(\frac{e^{-\beta_+\tau}\beta_+\beta_-}{\beta^2}\right)^j \times \tag{IV-6a}$$

$$D_1(j, \beta T_j) - D_1(j, \beta T_{j+1})\,e^{-(\beta_+\tau)} + D_2(j, \beta T_j)\,e^{-\beta T_j} - D_2(j, \beta T_{j+1})\,e^{-\beta T_{j+1} - \beta_+\tau} \quad .$$

For starting in the down (−) state:

$$P(t, -, \tau) = \sum_j \left(\frac{e^{-\beta_+\tau}\beta_+\beta_-}{\beta^2}\right)^j \times \tag{V-6b}$$

$$D_1(j, \beta T_j) + D_2(j, \beta T_j) - \frac{\beta_-}{\beta}[D_3(j, \beta T_{j+1}) - D_3(j, -\beta T_{j+1})\,e^{-\beta T_{j+1}}]$$

where T_j is a shifted time variable:

$$T_j = t - j\tau \quad ; \quad t > j\tau$$

$$= 0 \quad ; \quad t < j\tau \quad . \tag{V-6c}$$

It is important to note that terms containing T_j contribute only if $T_j > 0$; that is, a term such as $D_i(j, \beta T_j)$ carries with it an implicit step function $\Theta(T_j)$. The functions D_i are given by:

_____

[14] Note that these are not poles of the integrand before the expansion in powers of the exponential. A little thought shows that these expansions do make sense. For example, one can expand the simple integrand $1/(\omega - \omega_0)$ in powers of ω_0, changing the simple pole at ω_0 to a series of higher order poles at $\omega = 0$.

$$D_1(j, z) = \sum_{k=0}^{j} \frac{(2j-k-1)!}{k! \, (j-k)! \, (j-1)!} (-z)^k \quad , \quad j > 0;$$

<div align="right">(V-6d)</div>

$$= 1 \qquad\qquad , \quad j = 0.$$

$$D_2(j, z) = \sum_{k=0}^{j-1} \frac{(2j-k-1)!}{k! \, (j-k-1)! \, j!} z^k \quad , \quad j \geq 1 \qquad\qquad \text{(V-6e)}$$

$$D_3(j, z) = \sum_{k=0}^{j} \frac{(2j-k)!}{k! \, (j-k)! \, j!} (-z)^k \quad , \quad j \geq 0 \; . \qquad\qquad \text{(V-6f)}$$

This result, although intricate in form, is relatively simple to describe. One can describe the process of waiting for a clear interval as follows. Starting initially in the clear (up or +) state, the probability that the line of sight stays clear for a time τ is $e^{-\beta_+\tau}$. If the line of sight did not stay clear for the whole interval, it must have flipped to the blocked (down or −) state at some time before τ. One then must wait for the blocked state to flip and then begin the next clear interval. If the system flips before the time τ, then another cycle of waiting for the system to flip back to the (+) state and then counting down to τ begins. This alternation between waiting for the system to move from the (−) to (+) states and then testing the duration of the (+) state is responsible for the introduction of the step functions.

Although the complete expressions given in Eq. (V-6) are daunting, the resulting probabilities are not themselves complicated. Figure V-1 illustrates the probability of success (having a clear line of sight for an interval greater than or equal to τ) for $\beta_-\tau = 0.5$ and $\beta_+\tau = 0.5$. That is, the clear and obscured intervals both have an average value of 2τ. Two curves are shown; the higher curve, labeled P+, corresponds to the probability of success starting from the + (unobscured) state; the lower is the corresponding probability starting in the − (blocked) state. Both P+ and P− are identically zero for $t/\tau < 1$, since one cannot have an interval of length τ until at least that length of time has passed. P+ then jumps discontinuously to a value of $1 - e^{-\beta_+\tau}$, (≈ 0.61) corresponding to the probability that the line of sight has stayed continuously clear. On the other hand, P− is continuous at $t/\tau = 1$ since at least one flip (from − to +) must happen and therefore the probability increases smoothly from 0 at $t/\tau = 1$. Note that by $t/\tau = 4$, the probability of an interval of length at least τ is greater than 0.8. For long times, the distinction between P+ and P− becomes less important.
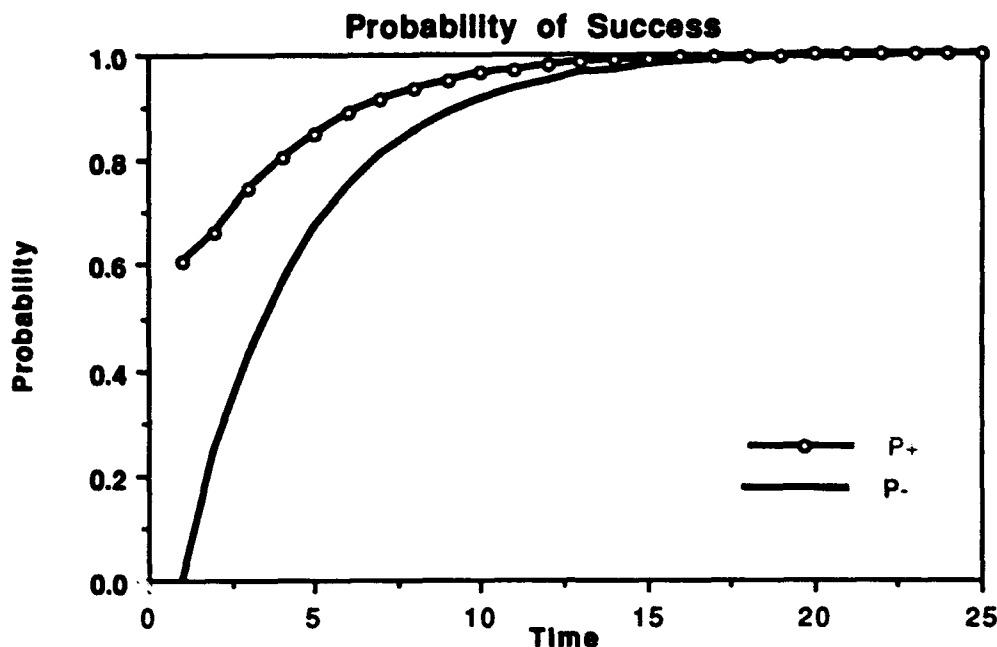
<div align="center">V-4</div>

Probability of Success

Figure V-1. Probability of having a clear interval of length $\geq \tau$ as a function of t/τ. ($\beta_-\tau$ = 0.5 and $\beta_+\tau$ = 0.5)

Figure V-2 shows the same curves for $\beta_-\tau = 0.5$ and $\beta_+\tau = 1.0$. For these values, the average duration of a clear interval is τ. There is a drop in the initial probability at $t/\tau =$ 1 to 0.31 (in comparison to the previous case) and the rate of increase of probability is slower. For 0.8 probability of success, one must have $t/\tau > 9$.

Figure V-3 shows the results for $\beta_-\tau = 0.5$ and $\beta_+\tau = 2.0$. In this case clear intervals last only $\tau/2$ on average; in consequence, the probability of success is less than 80% for $t/\tau < 25$.

Aside from the first discontinuity in P+ at $t/\tau = 1$, the curves are continuous. The step functions which appear in Eq. V-6 only introduce discontinuities in the higher order derivatives. Expanding the appropriate terms near the boundaries, one has:

$$D_1(j,z) + D_2(j,z)e^{-z} \propto z^{j+1}$$

$$(V-7)$$

$$D_3(j,z) - D_3(j,-z)e^{-z} \propto z^{j+1} \qquad z \ll 1, j > 0$$

so that only the (j+1) th derivatives exhibit the discontinuity.

V-5

Probability of Success

Figure V-2. Probability of having a clear interval of length $\geq \tau$ as a function of t/τ. ($\beta_-\tau = 0.5$ and $\beta_+\tau = 1.0$)

Probability of Success
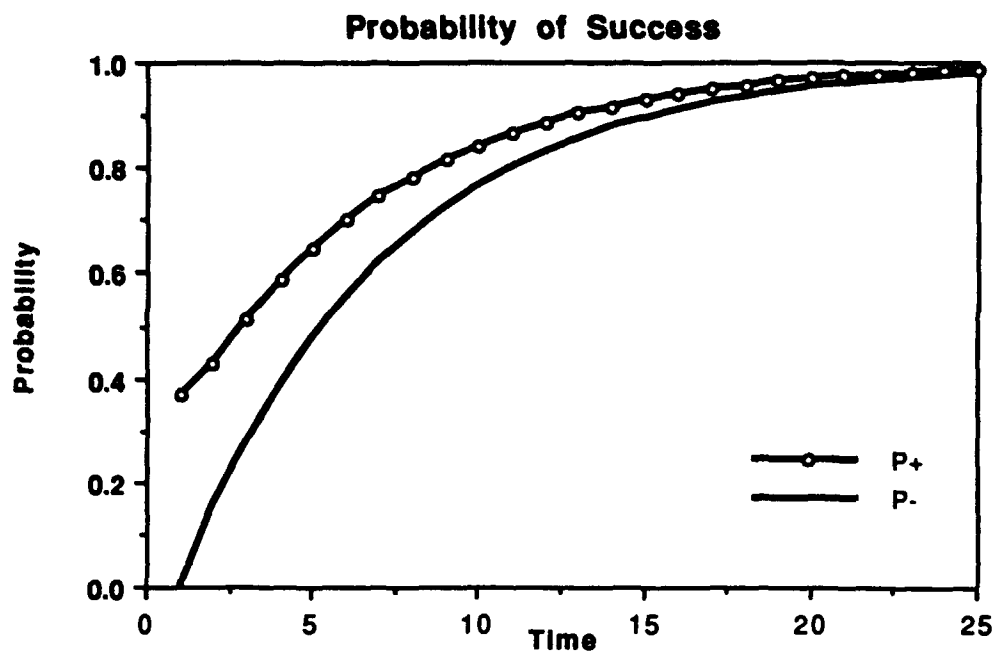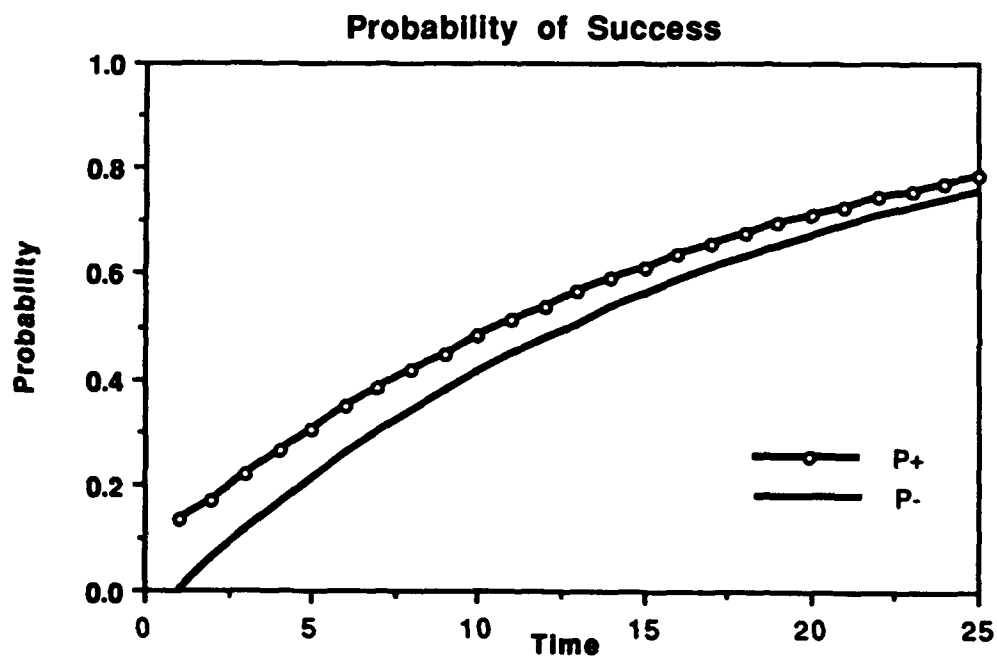

Figure V-3. Probability of having a clear interval of length $\geq \tau$ as a function of t/τ. ($\beta_-\tau = 0.5$ and $\beta_+\tau = 2.0$)

B. SINGLE POLE APPROXIMATION

Although the details of these results depend on the precise values of $\beta_-\tau$ and $\beta_+\tau$, it is clear that the overall structure is simple, particularly for longer times. The structure of the contour integral leads one to expect that, asymptotically, these probabilities will be dominated by a simple exponential form. In fact, these functions are well represented by a single pole approximation which looks for the smallest root of the denominators in Eqs. (V-5). Writing $\omega = i\,\kappa$, the transcendental equation that needs to be solved is

$$\kappa\,(\kappa - \beta) + \beta_+\beta_-\,e^{-(\beta_+ - \kappa)\tau} = 0 \ . \qquad (V-8)$$

There are two real roots for κ; the remaining (infinitely many) roots are complex. One root of this equation is always $\kappa = \beta_+$; however, an examination of Eq. (V-5) shows that both integrals have zero residue at that pole and do not contribute to the result. The other root must be determined numerically and the residue at that pole gives the asymptotic behavior desired. Denoting that root simply by κ, the single pole approximation is

$$P+ = 1 - \frac{(\kappa - \beta)\,(1 + \dfrac{\kappa\,(\kappa - \beta)}{\beta_+\beta_-})}{2\kappa - \beta - \tau\,\kappa\,(\kappa - \beta)}\,e^{-\kappa t}$$

$$P- = 1 - \frac{(\kappa - \beta)\,(1 - \dfrac{\kappa}{\beta_+})}{2\kappa - \beta - \tau\,\kappa\,(\kappa - \beta)}\,e^{-\kappa t} \ . \qquad (V-9)$$

In the special case of $\beta_+ - \beta_- + \beta_+\beta_-\tau = 0$, the two roots are degenerate and the correct asymptotic form is given by

$$P+ = 1 - \frac{\beta_-\tau}{1 + \beta_+\beta_-\tau^2/2}\,e^{-\beta_+ t}$$

$$P- = 1 - \frac{1 + \beta_-\tau}{1 + \beta_+\beta_-\tau^2/2}\,e^{-\beta_+ t} \ . \qquad (V-10)$$

Table V-1 gives the values of the transcendental root for a number of values of $\beta_+\tau$ and $\beta_-\tau$.

Table V-1. Values of the Dominant Pole

$\beta_-\tau \backslash \beta_+\tau$	0.1	0.2	0.5	1	2	5
0.1	0.0901	0.0811	0.0596	0.0358	0.0131	0.007
0.2	0.1792	0.1608	0.1169	0.698	0.0255	0.0013
.5	0.4404	0.3899	0.2760	0.1615	0.0588	0.0031
1.0	0.8590	0.7355	0.5	0.2857	0.1037	0.0057
2.0	1.5499	1.2798	0.8209	0.4568	0.1669	0.0098
5.0	2.6634	2.0683	1.2586	0.6935	0.2606	0.0172

The single pole approximation is actually very good for even moderate values of t/τ. Table V-2 gives the errors for $t/\tau = 0$, 1, and 2 for the cases given in Figs. V-1, V-2, and V-3. The errors are large at $t/\tau = 0$, moderate at $t/\tau = 1$ and negligible at $t/\tau = 2$. This indicates that the next (complex) root of Eq. (V-8) is usually large. An estimate of the location of that root has not been determined analytically but numerical searches show it to be typically 10 times greater than κ. Numerical experiments have not found any nontrivial cases (i.e., any case for which the probabilities are not essentially 0 or 1) for which the single pole approximation is not essentially perfect for $t/\tau > 2$.

Table V-2. Comparison of Exact Result and Single Pole Approximation

Parameters	t/τ	P+ Exact	P+ Approx	P– Exact	P– Approx
$\beta_+\tau = 0.5$	0	0	0.4145	0	– 0.307
$\beta_-\tau = 0.5$	1	0.6065	0.5557	0	0.0082
	2	0.6623	0.6629	0.2475	0.2474
$\beta_+\tau = 1.0$	0	0	0.2092	0	– 0.01681
$\beta_-\tau = 0.5$	1	0.3679	0.3272	0	0.0061
	2	0.4270	0.4275	0.1544	0.1543
$\beta_+\tau = 2.0$	0	0	0.0661	0	– 0.0583
$\beta_-\tau = 0.5$	1	0.1353	0.1195	0	0.0021
	2	0.1696	0.1697	0.0591	0.0591

Because the single pole approximation is so good, a practical use of the exact result would be to employ the exact result for small t/t (less than 2 or 3) and then use the single pole approximation. For convenience for such applications, the first few $D_k(j, z)$ are given below:

$$D_1(0, z) = 1$$

$$D_1(1, z) = 1 - z$$

$$D_1(2, z) = \frac{1}{2}[6 - 4z + z^2]$$

$$D_1(3, z) = \frac{1}{6}[60 - 36z + 9z^2 - z^3]$$

$$\text{(V-11a)}$$

$$D_2(0, z) = 0$$

$$D_2(1, z) = -1$$

$$D_2(2, z) = -(3 + z)$$

$$D_2(3, z) = -\frac{1}{2}(20 + 8z + z^2)$$

$$\text{(V-11b)}$$

$$D_3(0, z) = 1$$

$$D_3(1, z) = 2 - z$$

$$D_3(2, z) = \frac{1}{2}(12 - 6z + z^2)$$

$$D_3(3, z) = \frac{1}{6}(120 - 60z + 12z^2 - z^3) \ .$$

$$\text{(V-11c)}$$

VI. SUMMARY

The effects of obscuration on target acquisition and search have been modeled mathematically. In Section III, a statistical mechanical analog between the alternating states of clear and blocked lines of sight on the one hand and the up and down states of a ferromagnet was exploited to give a general relationship between the search timelines in the presence of obscuration and the free energy of a magnet (Eq. III-8). Applying this general result to the simplest magnet model (the Ising model) provides an explicit solution (Eq. IV-12). This choice leads to exponential distributions of the obscured and clear intervals (with characteristic exponents of β_- and β_+, respectively) and corresponds precisely to the Monte Carlo model of obscurants used by Kowalczyk and Rotman; the model parameters are determined by the average lengths of the obscured and clear intervals [$(\beta_-)^{-1}$ and $(\beta_+)^{-1}$] and the unobscured search time exponent, α. The probability of finding a target increases as:

$$P_{succeed}(t) = 1 - e^{-(\alpha - \beta_\alpha + \beta)t/2}[\cos^2\delta\theta + \sin^2\delta\theta e^{-\beta_\alpha t}] \qquad \text{(VI-1)}$$

where $\beta = \beta_- + \beta_+$ is the overall smoke correlation inverse time constant and

$$\beta_\alpha^2 = (\beta - \alpha\langle s\rangle)^2 + \alpha^2(1 - \langle s\rangle^2)$$

$$\cos^2\delta\theta = \frac{1 + \dfrac{\beta - \alpha\langle s\rangle}{\beta_\alpha}}{2}$$

$$\sin^2\delta\theta = \frac{1 - \dfrac{\beta - \alpha\langle s\rangle}{\beta_\alpha}}{2} \qquad \text{(VI-2)}$$

$$\langle s\rangle = \frac{\beta_- - \beta_+}{\beta}.$$

($\langle s\rangle$ is the "average spin" of the magnetic analog; the average visibility $\langle\eta\rangle = (1 + \langle s\rangle)/2$.) This provides explicit expressions for the numerical results of Kowalczyk and Rotman. It was also shown that this model could be equally well represented as a 3-state Markov process.

In Eq. (VI-1) the target acquisition task was modeled as a cumulative process for which the total time available on target was the relevant quantity, independent of how that time was accumulated or how many times the line of sight was interrupted by obscurants. This is consonant with the assumption that target acquisition is the consequence of a series of independent glimpses, each of which is of short duration compared to any change in the visibility conditions. Thus, the task is performed equally well when 10 seconds of clear viewing is provided whether it is provided in a single interval of 10 seconds or in 10 intervals of 1 second each. Another point of view would be to assume that a single continuous block of time is necessary to perform a task. This would be the case, for example, for a command-guided missile, but may also be relevant for other aspects of the target recognition process. The Ising-Markov model that was used to model the cumulative process can also be used to model the probabilities of having extended (continuous) intervals of clear line of sight to the target.

The result is algebraically complex in detail [Eq. (V-6)] and is represented by an infinite series of terms. However, numerically the results are relatively simple as illustrated in the figures (cf. Figs. V-1–V-3). Fortunately, a simple single term approximation to the exact expression can be derived. The probability of obtaining an interval of length τ is given by

$$P+ \ = \ 1 \ - \ \frac{(\kappa - \beta)\,(1 + \dfrac{\kappa\,(\kappa - \beta)}{\beta_+\beta_-})}{2\kappa - \beta - \tau\,\kappa\,(\kappa - \beta)}\,e^{-\kappa t} \qquad \text{(VI-3a)}$$

$$P- \ = \ 1 \ - \ \frac{(\kappa - \beta)\,(1 - \dfrac{\kappa}{\beta_+})}{2\kappa - \beta - \tau\,\kappa\,(\kappa - \beta)}\,e^{-\kappa t} \qquad \text{(VI-3b)}$$

where P+ gives the probability when the $t = 0$ state is known to be clear and P– gives the probability when the initial state is known to be obscured.

The parameter κ is determined by the root of the transcendental equation[15]:

$$\kappa\,(\kappa - \beta) + \beta_+\beta_-\, e^{-(\beta_+ - \kappa)\tau} \;=\; 0 \;.\qquad\qquad (VI\text{-}4)$$

This single term approximation is valid for "large" t/τ values; in practice, the error at $t/\tau = 2$ is negligible.

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[15] There are two real roots for κ; the remaining (infinitely many) roots are complex. One root of this equation is always β_+; the root to be employed in Eq. (V-2) is the other real root. In the special case of $\beta_+ - \beta_- + \beta_+\beta_-\tau = 0$, the real roots of Eq. (VI-3) are degenerate and a slightly different form is used:

$$P_+ \;=\; 1 \;-\; \frac{\beta_-\tau}{1 + \beta_+\beta_-\tau^2/2}\, e^{-\beta_+ t}$$

$$P_- \;=\; 1 \;-\; \frac{1 + \beta_-\tau}{1 + \beta_+\beta_-\tau^2/2}\, e^{-\beta_+ t}\;.$$